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The use of the constructive method in systems with static friction☆

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ABSTRACT

The problem of determining the accelerations in a mechanical system with friction at zero initial velocities is discussed. To approximate the solutions, an auxiliary system is constructed in which non-ideal geometrical constraints are produced by elastic forces, the structure of which complies with the static-friction law. It is proved that the auxiliary system is always solvable for the accelerations (possibly, non-uniquely) and hence a solution of the initial system also exists and can be constructed by taking a certain limit.

The idea of obtaining geometrical constraints by taking the limit of an elastic model with an unbounded increase in stiffness, previously attributed to Courant, found a rigorous proof in the case of ideal constraints, both bilateral[1] and unilateral.[2–5] An example of the effective use of this constructive approach is the theorem of existence for a system with an ideal unilateral constraint.[6] A similar approach was employed when realizing non-holonomic constraints by forces of viscous friction,[7,8] and also when investigating systems with sliding friction.[9,10]

1. Formulation of the problem and description of the method

In problems of mechanical system dynamics, the model of an absolutely rigid body is the simplest, and hence it is widely used when solving educational and scientific-research problems. We will consider a system M with generalized coordinates $\mathbf{q} \in \mathbf{R}^n$, subject to unilateral constraints

$$f_j(\mathbf{q}) \geq 0, \quad j = 1, \ldots, k \tag{1.1}$$

Formulae (1.1) express the condition that there should be no deformations when rigid bodies are in contact. We will assume that, at the instant of time t^0 considered, all relations (1.1) are satisfied as equalities, and all generalized velocities are equal to zero.

We will write the equations of motion in the form

$$\mathbf{A}\ddot{\mathbf{q}} = \mathbf{Q} + \sum_{j=1}^{k} \mathbf{R}_j \tag{1.2}$$

where $\mathbf{A}$ is the kinetic energy matrix, $\mathbf{Q}$ are generalized forces and $\mathbf{R}_j$ are the reactions of constraints (1.1).

We will initially assume that the constraints are ideal, in which case their reactions can be expressed by the formulae

$$\mathbf{R}_j = N_j \mathbf{n}_j, \quad \mathbf{n}_j = \nabla f_j / |\nabla f_j|, \quad N_j \geq 0, \quad j = 1, \ldots, k \tag{1.3}$$

To maintain the constraints when $t > t^0$ it is necessary to satisfy the relations

$$\ddot{f}_j = (\ddot{\mathbf{q}}, \mathbf{n}_j) \geq 0, \quad j = 1, \ldots, k \tag{1.4}$$

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In addition to this, the normal reactions N_j and normal accelerations $[\ddot{f}_j]$ cannot take non-zero values simultaneously. Altogether, the above constraints are described by the relations

$$N_j \geq 0, \quad \ddot{f}_j \geq 0, \quad N_j \ddot{f}_j = 0, \quad j = 1, \ldots, k \tag{1.5}$$

which are called the complementarity conditions or the Signorini conditions.

System M is described by relations (1.2), (1.3) and (1.5); as is well known from the linear complementarity theory,[11,12] it has a unique solution in the case when the matrix

$$\mathbf{B} = \|b_{ij}\|, \quad b_{ij} = (\mathbf{A}^{-1}\mathbf{n}_i, \mathbf{n}_j)$$

is positive definite. In the context of this consideration, this result is formulated as follows.

Theorem 1. The problem of determining the generalized accelerations in the case of ideal unilateral constraints with linearly independent normal vectors has a unique solution.

To construct the solution we can, in principle, consider all 2^k possibilities in formulae (1.5). For purpose's of economy, we can use one of the algorithms of linear programming – Lemke's method.[11,12]

The idea of a constructive approach, which goes back to d'Alembert, consists of replacing constraints (1.2) by elastic forces, generated by small deformations of the constraints, which occur with opposite sign of these inequalities. These forces will be calculated using the following formulae

$$N_j = c\delta_j, \quad \delta_j = \begin{cases} 0, & \text{if} \quad \ddot{f}_j \geq 0 \\ -(\ddot{\mathbf{q}}, \mathbf{n}_j), & \text{if} \quad \ddot{f}_j < 0 \end{cases}, \quad j = 1, \ldots, k \tag{1.6}$$

We will explain the meaning of relations (1.6). If there are no constraints (1.1), after a time τ, the coordinates $\mathbf{q}$ would have obtained an increment proportional to $\ddot{\mathbf{q}}$. If such a hypothetical displacement is not matched with the j-th constraint, it leads to its deformation by an amount $\delta_j\tau^2/2$. The factor c in formula (1.6) takes into account the value of τ and the stiffness; an unlimited increase in the latter (for a small but fixed value of τ) corresponds to taking the limit as $c \to \infty$.

We will define an auxiliary system M_c by Eq. (1.2), in which the reactions are calculated from formulae (1.3) and (1.6).

Theorem 2. System M_c has a unique solution $\ddot{\mathbf{q}}_c$, and it converts into the solution of system M as $c \to \infty$.

Proof. The quantity $\ddot{\mathbf{q}}_c$, which satisfies system M_c, is a stationary point of the constraint function

$$Z = \frac{1}{2}(\mathbf{A}\ddot{\mathbf{q}}, \ddot{\mathbf{q}}) - (\mathbf{Q}, \ddot{\mathbf{q}}) + \Pi, \quad \Pi = \frac{1}{2}c\sum_{j=1}^{k}\delta_j^2 \tag{1.7}$$

where Π is the potential energy of the deformations of the constraints. Since function (1.7) is strictly convex, this point is unique and is a global minimum.

Note that the potential energy of the system is equal to the work of the forces $\mathbf{Q}$ for small displacements (deformations) and is therefore limited. Consequently, as $c \to \infty$, the values of δ_i in formulae (1.6) tend to zero, i.e. relations (1.5) are satisfied in the limit.

Note that to construct the point $\ddot{q}_c$ we can use one of the well-known procedures for finding a global minimum of a convex function.

We will now consider the problem of solving system M when there is static friction. We will use the Amonton–Coulomb law in the form of accelerations of the form[13]

$$\mathbf{F}_j = -\mu_j \frac{\mathbf{w}_j}{|\mathbf{w}_j|} N_j, \quad |\mathbf{F}_j| \leq \mu_j N_j \tag{1.8}$$

where w_j is the tangential component of the acceleration at the corresponding contact point; these quantities are linear functions of $\ddot{\mathbf{q}}$, the coefficients of which are determined by the configuration of the system considered:

$$w_j = \sum_{s=1}^{n} t_{js}\ddot{q}_s, \quad j = 1, \ldots, k$$

The first formula of (1.8) corresponds to the beginning of sliding at the given point, while the second corresponds to the preservation of relative rest.

The solution of system (1.1), (1.2) with friction (1.8) is substantially more complicated than in the case of ideal constraints discussed above. It was shown in Ref. 14 that here there are examples of non-uniqueness of the solution. The sufficient conditions for existence and uniqueness were assumed in Ref. 15, and the criterion for the existence of a unique solution of this problem was later obtained in Ref. 16. It was also proved in Ref. 17 that the problem has a solution when a certain condition of the non-singularity system configuration is satisfied (this result is based on the properties of quasi-variational inequalities and their relation to non-linear complementarity problems).

The idea of a constructive approach in this case consists of using formulae (1.6) together with an approximation of the multivalued friction law (1.8) using of the following single-valued law with a variable friction coefficient

$$\mathbf{F}_j = -\chi(c|\mathbf{w}_j|)\mu_j \frac{\mathbf{w}_j}{|\mathbf{w}_j|} N_j, \quad \chi(x) = \begin{cases} x, & \text{if} \quad x < 1 \\ 1, & \text{if} \quad x \geq 1 \end{cases} \tag{1.9}$$

The auxiliary function $\chi(x)$ is continuous when $x \geq 0$, vanishes when $x = 0$ and is equal to unity outside the interval $(0, c^{-1})$, which is confined to the origin of coordinates when $c \to \infty$.

Example. A rigid body of unit mass and weight is at rest on a rough horizontal plane. At a certain instant of time a horizontal force $Q > 0$ begins to act on it. It is required to determine the acceleration $\ddot{q}$.

The solution for friction law (1.8) is easily obtained by exhaustive search:

$$\ddot{q} = \max\{0, Q - \mu\} \tag{1.10}$$

The use of approximation (1.9) leads to the equation

$$\ddot{q} = Q - \mu\chi(c\ddot{q})$$

the left-hand side of which increases monotonically in $\ddot{q}$ while the right-hand side decreases monotonically. The sole solution of this equation is described by the formula

$$\ddot{q} = \begin{cases} Q - \mu, & \text{if} \quad Q > \mu + c^{-1} \\ \dfrac{Q}{1 + c\mu}, & \text{if} \quad Q < \mu + c^{-1} \end{cases} \tag{1.11}$$

It is obvious that formulae (1.11) reduce to expression (1.10) in the limit as $c \to \infty$.

2. Systems with associated configurations

In the example given above, the normal reaction is equal to the weight, irrespective of which body is at rest or sliding on the support. The presence of this property considerably simplifies the analysis of the system;[13,18] we will formulate it in general form.

Definition. We will say that a system configuration is associated if the normal reactions of the constraints are uniquely defined by the values of the generalized forces, irrespective of the friction law.

Theorem 3.[18] In a system with an associated configuration and friction law (1.8) the actual motion is uniquely defined by the condition for a minimum of the constraint

$$Z(\ddot{\mathbf{q}}) = \frac{1}{2}(A\ddot{\mathbf{q}}, \ddot{\mathbf{q}}) - (\mathbf{Q}, \ddot{\mathbf{q}}) - W, \; W = -\sum_{j=1}^{k} \mu_j N_j |w_j| \tag{2.1}$$

To prove this, it is sufficient to note that the function (2.1) is strictly convex, and hence its stationary point is a global minimum. The quantity W is equal to the work of the friction forces. Consequently, the gradient of the function Z is expressed by the formula

$$\nabla Z(\ddot{\mathbf{q}}) = \mathbf{A}\ddot{\mathbf{q}} - \mathbf{Q} - \nabla W$$

where, by the definition of generalized forces, the quantity ∇W is equal to the total friction force in Eq. (1.2). Hence, this equation is equivalent to the condition for the constraint to be steady.

Examples *(1°).* In the system of the previous section we have $w = \ddot{q}$ and the function (2.1) takes form

$$Z = \frac{1}{2}\ddot{q}^2 - Q\ddot{q} + \mu \mid \ddot{q} \mid \tag{2.2}$$

As can easily be shown the point (1.11) gives a minimum of function (2.2).

2°. We will complicate the previous example by constructing a "pyramid" of k heavy rectangular blocks with masses m_j, each of which is acted upon by a horizontal force Q_j ($j = 1, \ldots, k$). The solution, by exhaustive search of rectilinear inspection, requires the consideration of 3^k possibilities. Lemke's method requires a multinominal volume of calculations, but is extremely lengthy to describe.[12] According to Theorem 3, to solve it we can use one of the methods of convex analysis to find a global minimum of the function

$$Z = \frac{1}{2}\sum_{j=1}^{k}\left(m_j\ddot{q}_j^2 - Q_j\ddot{q}_j\right) + g\sum_{j=1}^{k}\mu_j M_j|\ddot{q}_{j+1} - \ddot{q}_j|, \quad M_j = \sum_{s=j}^{k} m_j \tag{2.3}$$

Note that the procedure described can also be used when the vectors Q_j are non-collinear, in which the vectors 92d can have arbitrary directions (in the horizontal plane).

We will investigate a system with elastic forces (1.9) and an associated configuration. As can be verified, in this case the work of the friction forces is expressed by the formula

$$W_c = -\sum_{j=1}^{k}\mu_j N_j\left(\chi(c|w_j|)|w_j| - \frac{1}{2c}\chi^2(c|w_j|)\right) \tag{2.4}$$

We can further compile a constraint function Z_c by analogy with condition (2.1) but with W replaced by W_c. The minimum of this function defines the motion of an auxiliary system. By noting that $W_c \to W$ when $c \to \infty$ and for any w, we arrive at the following assertion.

Theorem 4. The actual motion in system M with associated configuration and friction law (1.8) can be defined as the limit of the actual motion of the auxiliary system M_c with friction (2.9) as $c \to \infty$.

Remark. This result has a more illustrative than practical value, since the solution of the auxiliary system is not simpler than the solution of the initial system (in both cases it is required to minimize the constraint function).

3. The existence theorem in systems with non-associated configuration

We will now consider the general case when the normal reactions of the constraints cannot be uniquely determined from the contact conditions without taking shear stresses into account. We will use formulae (1.6) and (1.9) to construct the auxiliary system. We will formulate the main property of the contact stresses obtained in this model.

The dissipation property of elastic reactions. For any vector of the generalized accelerations $\ddot{\mathbf{q}} \in \mathbb{R}^n$ the overall reaction, calculated from formulae (1.6) and (1.9), satisfies the inequality

$$(\ddot{\mathbf{q}}, \mathbf{R}(\ddot{\mathbf{q}})) \le 0 \tag{3.1}$$

To prove this we will assume that the system undergoes a small displacement in a time Δt from the equilibrium position with a constant acceleration $\ddot{\mathbf{q}}$. As a result, some of inequalities (1.1) will break down, and the reactions of the remaining constraints will be equal to zero. At each of the contact points, for which the normal displacement is negative, the normal component of the reaction will be positive, in view of formula (1.6); in this case, as a consequence of definition (1.9), the tangential displacement and the tangential reaction will be anticollinear. Hence inequality (3.1) follows.

Theorem 5. Equations (1.2) with elastic reaction forces (1.6) and (1.9) are solvable for any vector of the generalized forces $\mathbf{Q} \in \mathbb{R}^n$.

Lemma 1. Suppose the operator $P : \bar{S}_a \mapsto \mathbb{R}^n$ is continuous in a sphere $\bar{S}_a = \left\{x \in \mathbb{R}^n \,|\, |x| \le a\right\}$, and the following inequality is satisfied at the boundary points

$$(P(\mathbf{x}), \mathbf{x}) > 0 \tag{3.2}$$

Then, in the sphere S_a, we obtain a point $\mathbf{x}^*$, at which $P(\mathbf{x}) = 0$.

Proof. In view of the continuity, the operator P is bounded, i.e., $\|P(\mathbf{x}) \le M\|$. Moreover, for a sufficiently small number $\varepsilon > 0$, in the boundary layer $a - \varepsilon \le \|\mathbf{x}\| \le a$ the following condition is satisfied

$$(P(\mathbf{x}), \mathbf{x}) > \varepsilon \tag{3.3}$$

Consider the auxiliary operator

$$U(\mathbf{x}) = \mathbf{x} - \lambda P(\mathbf{x})$$

It is continuous, and for sufficiently small values of λ it maps the sphere $\bar{S}_a$ into itself. In fact, if $\|\mathbf{x}\| \le a - \varepsilon$, the following inequality is satisfied

$$\|U(\mathbf{x})\| \le \|\mathbf{x}\| + \lambda\|P(\mathbf{x})\| \le a - \varepsilon + \lambda M$$

and when $a - \varepsilon \le \|\boldsymbol{x}\| \le a$ we have the limit

$$\|U(\mathbf{x})\|^2 = \|\mathbf{x}\|^2 - 2\lambda(P(\mathbf{x}), \mathbf{x}) + \lambda^2\|P(\mathbf{x})\|^2 < a^2 - 2\lambda\varepsilon + \lambda^2 M^2$$

Assuming

$$\lambda = \min\left\{\frac{\varepsilon}{M}, \frac{2\varepsilon}{M^2}\right\}$$

we arrive at the inequality

$$\|U(\mathbf{x})\| \le a$$

By the well-known Brourver's theorem, the continuous mapping $U(\mathbf{x})$, that maps the sphere S_a into itself, has a fixed point, which is also the required zero of the operator P($\mathbf{x}$).

Lemma 2. Suppose the operator $T(\mathbf{x})$ is continuous in $\mathbb{R}^n$ and the following estimate holds

$$(T(\mathbf{x}), \mathbf{x}) \ge \gamma(\|\mathbf{x}\|)\|\mathbf{x}\| \tag{3.4}$$

for a certain positive function $\gamma(||\mathbf{x}||)$, when $\gamma(||\mathbf{x}||) \to \infty$ as $||\mathbf{x}|| \to \infty$. Then the equation $T(\mathbf{x}) = \mathbf{y}$ is solvable for any $\mathbf{y} \in \mathbb{R}^n$.

Proof. We put $\mathrm{P}(\mathbf{x}) = \mathrm{T}(\mathbf{x}) - \mathbf{y}$, in which case the problem reduces to finding the zero of the operator P($\mathbf{x}$). Since

$$(P(\mathbf{x}), \mathbf{x}) = (T(\mathbf{x}), \mathbf{x}) - (\mathbf{y}, \mathbf{x}) \ge \|\mathbf{x}\|(\gamma(\|\mathbf{x}\|) - \|\mathbf{y}\|)$$

the assertion of Lemma 2 follows from Lemma 1.

Proof of Theorem 5. We will represent Eq. (1.2) in the form

$$T(\ddot{\mathbf{q}}) = Q, \quad T(\ddot{\mathbf{q}}) = A\ddot{\mathbf{q}} - R(\ddot{\mathbf{q}})$$

In view of the proved dissipation property of the reactions, expressed by formulae (1.6) and (1.9), the following estimate holds

$$(T(\ddot{\mathbf{q}}), \ddot{\mathbf{q}}) \ge (\mathbf{A}\ddot{\mathbf{q}}, \ddot{\mathbf{q}}) \ge \lambda_m\|\ddot{\mathbf{q}}\|^2$$

where $\lambda_m > 0$ is the minimum eigenvalue of the kinetic energy matrix $\mathbf{A}$. The assertion of the theorem follows from Lemma 2.

In the case of a non-associated friction law, the solution of non-linear system (1.2), (1.6), (1.9) may not be unique. We will illustrate this by a model example.

Example. The system

$$\begin{Vmatrix}1 & 1\\ 1 & 2\end{Vmatrix}\ddot{\mathbf{q}} = \begin{Vmatrix}-2\\ -1\end{Vmatrix} + \begin{Vmatrix}F\\ N\end{Vmatrix} \qquad (3.5)$$

with a "rigid" constraint $q_2 \geq 0$ has the solutions $\ddot{q}_1 = -3,\ \ddot{q}_2 = 1,\ F = N = 0$ (detachment), $\ddot{\mathbf{q}} = 0, F = 2, N = 1$ (rest) and $\ddot{q}_1 = N - 1 < 0,\ \ddot{q}_2 = 0,\ N = (\mu - 1)^{-1},\ F = \mu N$ (slippage), where the second and third solutions are only possible when $\mu > 2$. We then use formulae (1.6) and (1.9) to calculate the reactions, assuming $w = \ddot{q}_1$, $\delta = \max\{0, -\ddot{q}_2\}$. Instead of system (3.5) we obtain the following non-linear system in $\ddot{q}$:

$$\begin{Vmatrix}1 & 1\\ 1 & 2\end{Vmatrix}\ddot{\mathbf{q}} = \begin{Vmatrix}-2\\ -1\end{Vmatrix} + c\begin{Vmatrix}-\mu\chi(c\,|\,\ddot{q}_1\,|)\operatorname{sign}\ddot{q}_1\\ 1\end{Vmatrix}\max\{0, -\ddot{q}_2\} \qquad (3.6)$$

In the interval $\ddot{q}_2 \geq 0$ (detachment) the solutions of both systems are identical. In the region $\ddot{q}_1 < 0,\ \ddot{q}_2 < 0$, when $\mu > 2$ we have the following solution for fairly large values of c

$$\ddot{q}_1 = -1 + \frac{c+2}{c(\mu-1)-1}, \quad \ddot{q}_2 = -\frac{1}{c(\mu-1)-1}$$

corresponding to slippage, while in the region $\ddot{q}_1 \geq 0,\ \ddot{q}_2 < 0$ we have the solution

$$\ddot{q}_1 = \frac{3}{c-1}, \quad \ddot{q}_2 = -\frac{1}{c-1}$$

corresponding to a state of rest.

Theorem 6. Equations (1.2) with constraints (1.1) and static friction (1.8) are solvable for any vector of the generalized forces $\mathbf{Q} \in \mathbb{R}^n$.

Proof. We will take an arbitrary sequence of positive numbers $\{c_j \to +\infty\}$. For each of these we construct an auxiliary system (1.2), (1.6), (1.9) and obtain some solution $\ddot{\mathbf{q}}_{(j)}$ of it (the existence of a solution is guaranteed by Theorem 5). We will show that the sequence $\{\ddot{\mathbf{q}}_{(j)}\}$ is bounded in $\mathbb{R}^n$. By virtue of inequality (3.1) (the dissipation property of of the reaction) for any vector $\ddot{\mathbf{q}} \in \mathbb{R}^n$ the following relation holds

$$(A\ddot{\mathbf{q}}, \ddot{\mathbf{q}}) - (\mathbf{Q}, \ddot{\mathbf{q}}) \leq 0$$

whence the following limit holds

$$\|\ddot{\mathbf{q}}\| \leq \lambda_m^{-1}\|\mathbf{Q}\|$$

which indicates the boundedness of the sequence $\{\ddot{\mathbf{q}}_{(j)}\}$.

By the Bolzano-Weierstrass theorem, the bounded sequence $\{\ddot{\mathbf{q}}_{(j)}\}$ has at least one limit point $\ddot{\mathbf{q}}^*$. We will prove that the vector $\ddot{\mathbf{q}}^*$ satisfies inequalities (1.1), where the values of the reactions corresponding to it agree both with the equations of motion (1.2) and with the friction law (1.8).

We will show that the inequality $f(\ddot{\mathbf{q}}^*) < 0$ is impossible. In fact, in this case the sequence $f(\ddot{\mathbf{q}}_{j,k})$ should have a non-zero limit, and the normal reaction in formula (1.6) should be infinitely large. This contradicts the formula

$$(\mathbf{A}\ddot{\mathbf{q}}, \ddot{q}) - (\mathbf{Q}, \ddot{\mathbf{q}}) = (\mathbf{R}, \ddot{\mathbf{q}})$$

in which the left-hand side is bounded.

We will assume that the strict inequality $f(\ddot{\mathbf{q}}*) > 0$ is satisfied, in which case all the terms of the sequence $f(\ddot{\mathbf{q}}_{j,k})$, beginning with a certain number, are also positive, and in formulae (1.6) $N = 0$, which agrees with the complementarity conditions (1.5).

In the case of a non-zero slip velocity at one of the contacts $\mathbf{w}^* \neq 0$, the corresponding terms of the sequence $\mathbf{w}_{j,k}$ will also be separted from zero, and hence $c|\mathbf{w}_{j,k}| \to \infty$, and $\chi(c|\mathbf{w}_{j,k}| = 1)$ for sufficiently large values of the subscript. This indicates that the first formula of (1.8), corresponding to slippage, is satisfied.

The remaining case is when the inequality $\chi(c|\mathbf{w}_{j_k}| < 1)$ is satisfied for all terms of the sequence. Hence it follows that $\mathbf{w}^* = 0$, which agrees with the second formula of (1.8), describing static friction. The theorem is completely proved.

Remarks 1°. The existence theorem was proved earlier for certain additional restrictions[17] using methods of the theory of quasi-variational inequalities. The constructive method used in this paper does not involve these restrictions.

2°. The non-uniqueness of the solution of the auxiliary system for values of c as large as desired confirms the non-uniqueness of the solution of the initial system (see the example given above). However, as the following example shows, we cannot assert that every solution of the initial system can be obtained in this way.

Example. A uniform rod OC of length l is hinged to a fixed support, and its upper end is in contact with the lower surface of a rough horizontal beam (see the figure). The system M has a single (rotational) degree of freedom and is described by the relations

$$m\rho^2\ddot{\varphi} = \frac{1}{2}lP\sin\varphi + lN\sin\varphi - lF\cos\varphi, \quad \varphi \geq \varphi_0, \quad \varphi(t_0) = \varphi_0, \quad \dot{\varphi}(t_0) = 0$$

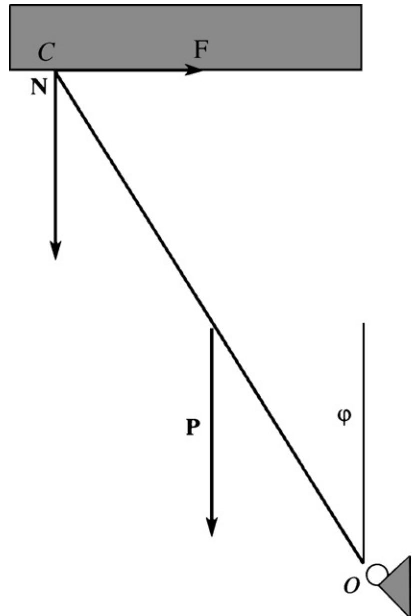

Fig. 1.

with the unilateral constraint $\varphi \geq \varphi_0$, where the value φ_0 corresponds to contact. When $N = F = 0$ Eq. (3.7) has a solution for which $N = F = 0$, $\ddot{\varphi} > 0$, i.e., the rod breaks away from the support. On the other hand, when the following inequality is satisfied

$$\mu > \mathrm{tg}\varphi_0$$

the rod can remain in equilibrium, in which case the components of the reaction are related by the expressions

$$N \geq \frac{1}{2\mu\mathrm{ctg}\varphi_0 - 1}P, \quad F = \left(N + \frac{1}{2}P\right)\mathrm{tg}\varphi_0 \tag{3.7}$$

The auxiliary system M_c has the unique solution $\ddot{\varphi} > 0$, since, when $\ddot{\varphi} < 0$, it follows from formulae (1.9) that $F < 0$, which does not agree with formulae (3.7).

Note that this lack of agreement is due to the fact that there is a strict relation between the normal and tangential deformations. It can be removed by increasing the dimension of the auxiliary system by taking into account linear or flexural deformations of the rod.

4. Discussion of the results

As follows from Theorem 5, the problem of determining the generalized accelerations and reactions of the constraints in a system of rigid bodies with dry static friction is always solvable. However, the solution may be non-unique, i.e., in addition to equilibrium conservation the start of motion is possible (slippage or detachment). This non-uniqueness is not related directly to the use of the model of an absolutely rigid body: it is present in equal measure in more complex contact models, which take small deformations into account. We can conclude that the reason for the uncertainty has different origins.

The uncertainty is not removed nor the failure of Amonton–Coulomb formula (1.8) to describe static friction if only the limit values of the friction force is non-zero. To justify this thesis we will again consider the system shown in the figure. It is obvious that weakening of the constraint is possible for as small a value of the angle $\varphi_0 > 0$ as desired (including the presence of adhesion, if the weight is sufficiently great). At the same time equality (3.7) is also satisfied for a fairly small value of this angle. Consequently, the uncertainty is not removed.

It was suggested in Ref. 14 that, at the instant of time considered, the normal reactions of the constraints be specified. This is equivalent to assuming that active forces are applied to the system which reduce these reactions and which vanish at a given moment. If it turns out that, for such values of the reactions, there are friction forces which maintain the system in a state of rest, this state can be recognised as being real. This rule is also not universal, since the arbitrarily specified reactions may not correspond to any of the possible solutions of system (1.2).

In specific systems, it is possible to select a "true" motion from several possible motions using the stability criterion.[9] However, in general, there may be several stable motions.[10] A similar situation also arises when there is no friction in the classical Euler example of the "buckling" of a loaded column.

A criterion of the existence of a unique solution of this problem was obtained earlier in Ref. 16. In the general case, when there are several solutions, methods of choosing them require further investigation.

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